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LETTER TO THE EDITOR

Hierarchical percolation model with anomalous multifractal measure

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Abstract. Hierarchical models are introduced to discuss the anomalous multifractal measure of the current fraction distribution on the percolating cluster analytically. The deterministic fractal model proposed by de Arcangelis, Redner and Coniglio is extended to take into account the faster decreasing minimum current fraction than power law. It is shown that, in the hierarchical models constructed from the generator of a ladder configuration with m plaquettes, the minimum current fraction has the dependence on L, $i_{\min} \approx \exp[-cm(\ln L)^2]$, where L is the system size and c a constant. The other hierarchical model showing $i_{\min} \approx \exp[-c(\ln L)\{\ln(\ln L)\}]$ is also found.

Recently, there has been increasing interest in the critical behaviour of random resistor networks. It has been found that electrical properties of self-similar resistor networks should be characterized by an infinite set of exponents (Rammal *et al* 1985a, b, de Arcangelis *et al* 1985a, b, 1986). The multifractal structure of the current distribution has been studied (Nagatani 1987, Fourcade and Tremblay 1987, Blumenfeld *et al* 1987, Nagatani *et al* 1989). In many cases, specific members of families of fractal dimensions represent geometrical and physical substructures of the underlying self-similar structure. The fact that an infinite set of exponents is necessary to characterize completely the properties of self-similar resistor networks has analogues in most fields, such as turbulence, diffusion-limited aggregation, localization and dynamical system.

Very recently, breakdown of multifractal behaviour in a range of negative moments has drawn much attention (Blumenfeld *et al* 1986, 1987, Fourcade and Tremblay 1987, Lee and Stanley 1988, Blumenfeld and Aharony 1989, Kahng and Lee 1990, Mandelbrot *et al* 1990, Stanley *et al* 1990). The breakdown phenomenon is due to the dependence of the smallest current fraction upon the size L. There has been much discussion of measures such that the partition function diverges faster than a power law, for small enough negative q values. There exist two recently proposed forms for the dependence on L of $i_{\min}$, the smallest of all the current fraction.

(i) Blumenfeld *et al* (1987) proposed that $i_{\min}$ decreases exponentially with size L,

$$i_{\min}(L) \approx \exp(-cL^x). \tag{1a}$$

(ii) Fourcade and Tremblay (1987) proposed the dependence of $i_{\min}$ on L

$$i_{\min}(L) \approx \exp[-c(\ln L)^2]. \tag{1b}$$

The 'free energy' $\tau(q)$ is singular at $q=0$ and fails to be defined for $q<0$ because of faster decreasing minimum current fraction than a power law. Mandelbrot (1990)

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named the 'left-sided' multifractality for the 'anomalous' multifractal measures. The behaviour of the smallest current fraction is very sensitive to the geometry of the cluster. This manifests itself in a large degree of variability in the minimum current fraction on different clusters of the same size.

In this letter, we present some hierarchical percolation models in order to discuss the anomalous multifractal measure of the current fraction distribution on the percolating cluster analytically. We extend the deterministic fractal model proposed by de Arcangelis *et al* (1985) to take into account the faster decreasing minimum current fraction than power law. We show that the minimum current fraction scales as $i_{\min} \approx \exp[-cm(\ln L)^2]$ in the hierarchical model constructed from the generator of a ladder configuration with m plaquettes. We also find the hierarchical model with $i_{\min} \approx \exp[-c(\ln L)\{\ln(\ln L)\}]$.

Let us construct the deterministic fractal model to mimic the infinite cluster at the percolation threshold. The de Arcangelis–Redner–Coniglio model can predict behaviour of a typical cluster (the positive moment of the current fraction has a power law). However, it cannot predict behaviour of a rare cluster such that the minimum current fraction decreases faster than a power law. We introduce a generator for the rare cluster into the typical cluster. We construct the deterministic fractal model by using two types of generators: one is the generator for the typical cluster and the other is the generator for the rare cluster. In order to imitate an infinite cluster, we choose the generator which has the typical values of the scaling exponents of the infinite cluster: the fractal dimensions of the backbone and the cutting bonds and the scaling exponent of the conductivity. Figure 1 shows the initiator and generators. Figure 1(a) indicates the initiator, figure 1(b) represents the generator for the typical cluster and figure 1(c), (d) and (e) show the generators for the rare cluster. The method of constructing the deterministic fractal proceeds iteratively. The first generation is obtained from the initiator (figure 1(a)) by replacing each bond with each generator:

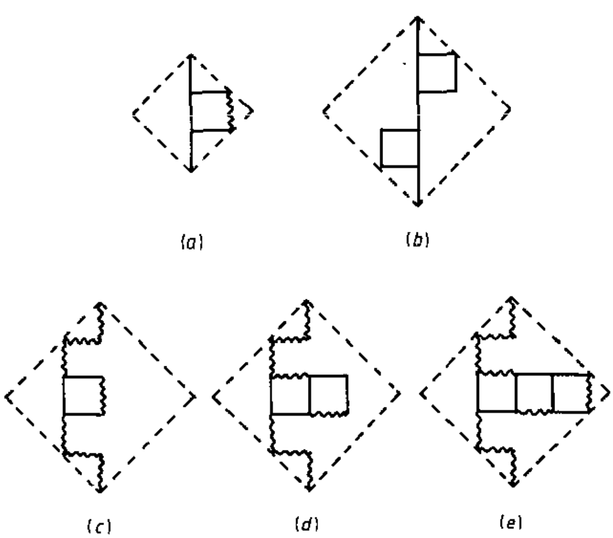


Figure 1. The initiator and the generators. (a) The initiator. (b) The generator for the typical cluster. (c) The generator with a single plaquette for the rare cluster. (d) The generator with two plaquettes for the rare cluster. (e) The generator with three plaquettes for the rare cluster.

the solid lines are replaced by the generator (b) and the wavy line is replaced by the generator (c) (or generator (d) or generator (e)). The length scale is transformed by the factor $b = 5$. The second generation is obtained from the first generation by replacing each bond with each generator. The resultant system is scaled up to five times. The process is continued *ad infinitum*. In this way one can obtain the deterministic fractal model. We call the solid line the typical bond and the wavy line the rare bond. In the deterministic fractal for the typical cluster, the fractal dimensionalities of the backbone and the cutting bonds are $d_b = \log 11/\log 5$ $(=1.490)$ and $d_c = \log 3/\log 5$ $(=0.683)$ respectively, close to the known percolation values ($d_b = 1.62$ and $d_c = 0.75$) (Stauffer 1985). The values in our model are less poor approximations than those of the model of de Arcangelis *et al* (1985). This is due to the choice of the generator with m plaquettes for the rare cluster. Also we choose the scale factor $b = 5$ to embed three plaquettes into the square. By using the generator with larger scale factors, one can obtain deterministic fractal model with more accurate scaling exponents. However, we choose the smallest scale factor $b = 5$ for simplicity. The exponent, describing the power law dependence on scale length L of the conductivity $L^{-t/v}$ is given by $t/v = \log \frac{9}{2}/\log 5$ $(=0.9345)$. This is a good approximation of an accurate value 0.97 obtained by computer simulation (Stauffer 1985).

Firstly, we consider the fractal model constructed from the generators (b) and (c). Let us consider the renormalization of the resistance at the nth generation (see figure 2). The resistances of the typical bonds between the $(n-1)$th and the nth generations are related by the recursion relation

$$R_{a,n-1} = \tfrac{9}{2} R_{a,n} \tag{2}$$

where $R_{a,n-1}$ and $R_{a,n}$ are the resistances of the typical bonds at the $(n-1)$th and nth generations. The value $\frac{9}{2}$ is obtained from the total resistance of the generator (b) when the resistance of a single bond is a unit value. The scaling exponent of the conductivity $L^{-t/v}$ is given by $t/v = \log R/\log b = \log \frac{9}{2}/\ln 5$ $(=0.935)$, very close to the known percolation value 0.97. Similarly, the resistances of the rare bonds at the $(n-1)$th and nth generations are related by

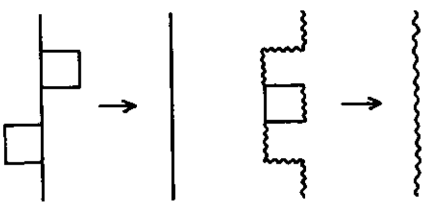
$$R_{b,n-1} = 6R_{b,n} + R_{a,n}[2 + (R_{b,n}/R_{a,n})]/[3 + (R_{b,n}/R_{a,n})] \tag{3}$$

where $R_{b,n-1}$ and $R_{b,n}$ are the resistances of the rare bonds at the $(n-1)$th and nth generations. The ratio of resistances of typical and rare bonds is given by

$$(R_{b,n-1}/R_{a,n-1}) = \tfrac{4}{3}(R_{b,n}/R_{a,n}) + \tfrac{2}{9}[2 + (R_{b,n}/R_{a,n})]/[3 + (R_{b,n}/R_{a,n})]. \tag{4}$$

By renormalizing the Nth generation $(N-n)$ times, the resistance ratio scales as

$$(R_{b,n}/R_{a,n}) \approx (\tfrac{4}{3})^{N-n}. \tag{5}$$



(a) (b)

Figure 2. Renormalization process. (a) The generator for the typical cluster is renormalized to the typical bond. (b) The generator for the rare cluster is renormalized to the rare bond. The typical bond and the rare bond are indicated by the solid line and the wavy line.

We can obtain the same scaling form (5) of the resistance ratio for the generators (d) or (e). This is due to having the same number of the cutting bonds in the generators (c), (d) and (e) (figure 1).

We consider the current fractions in the ladder configuration with m plaquettes (figure 3). The current fractions $i_{a,n}$ and $i_{b,n}$ flowing through the bonds at the nth generation in figure 3(a) are given by

$$i_{a,n} = R_{a,n}^{-1}/\{R_{a,n}^{-1} + (2R_{a,n} + R_{b,n})^{-1}\} \tag{6a}$$

$$i_{b,n} = (2R_{a,n} + R_{b,n})^{-1}/[R_{a,n}^{-1} + (2R_{a,n} + R_{b,n})^{-1}]. \tag{6b}$$

In the limit of $N \to \infty$ and $n \to 0$,

$$i_{a,n} \to 1 \qquad i_{b,n} \approx (R_{a,n}/R_{b,n}) \to 0. \tag{7}$$

Similarly, the current fractions $i_{a,n}$, $i_{b,n}$ and $i_{c,n}$ in figure 3(b) are obtained. In the limit of $N \to \infty$ and $n \to 0$,

$$i_{a,n} \to 1 \qquad i_{b,n} \approx (R_{a,n}/R_{b,n}) \to 0 \qquad i_{c,n} \approx (R_{a,n}/R_{b,n})^2 \to 0. \tag{8}$$

Also we obtain the current fractions $i_{a,n}$, $i_{b,n}$, $i_{c,n}$ and $i_{d,n}$ in figure 3(c). In the limit of $N \to \infty$ and $n \to 0$,

$$\begin{aligned} &i_{a,n} \to 1 \qquad i_{b,n} \approx (R_{a,n}/R_{b,n}) \to 0 \\ &i_{c,n} \approx (R_{a,n}/R_{b,n})^2 \to 0 \qquad i_{d,n} \approx (R_{a,n}/R_{b,n})^3 \to 0. \end{aligned} \tag{9}$$

We can find that the minimum current fraction $i_{\min,n}$ in the ladder configuration with m plaquettes at the nth generation scales as

$$i_{\min,n} \approx (R_{a,n}/R_{b,n})^m. \tag{10}$$

The minimum current fraction $i_{\min}$ for the hierarchical model at the Nth generation is given by a multiplicative process of minimum current fraction $i_{\min,n}$ in each stage

$$\begin{aligned} i_{\min} &\approx i_{\min,1} i_{\min,2} \ldots i_{\min,n} \ldots i_{\min,N} \\ &\approx (R_{a,1}/R_{b,1})^m (R_{a,2}/R_{b,2})^m \ldots (R_{a,n}/R_{b,n})^m \ldots (R_{a,N}/R_{b,N})^m. \end{aligned} \tag{11}$$

By using (5) we obtain

$$i_{\min} \approx \exp[-m \ln \tfrac{4}{3}/\{2(\ln 5)^2\}(\ln L)^2] \tag{12}$$

where we used the relationship $L = 5^N$. We can find the minimum current fraction decreasing faster than power law. The minimum current fraction depends strongly on the number of plaquettes. The result with $m = 1$ agrees with that of Fourcade and Tremblay (1987).

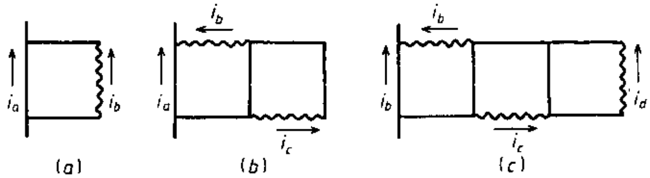


Figure 3. The current fractions in the ladder configuration with m plaquettes. (a) $m = 1$. (b) $m = 2$. (c) $m = 3$.

We consider the partition function $\Sigma\, i^q$. For $q \geqslant 0$, in a sufficiently large N limit, the partition function scales as

$$\sum_i i^q \approx \{3 \cdot 1^q + 2 \cdot (\tfrac{3}{4})^q + 6 \cdot (\tfrac{1}{4})^q\}^N. \tag{13}$$

Finally, we obtain the following expression of $\tau(q) = (q-1)D(q)$ for $q \geqslant 0$

$$(q-1)D(q) = -\ln[3 + 2(\tfrac{3}{4})^q + 6(\tfrac{1}{4})^q]/\ln 5 \tag{14}$$

where $\Sigma\, i^q = L^{-\tau(q)}$.

The current fraction in the generator of the typical cluster contributes dominantly to the partition function for $q \geqslant 0$. On the other hand, for $q < 0$, the minimum current fraction $i_{\min}$ dominates the partition function. Therefore, the partition function for $q < 0$ cannot scale. We can obtain the analytical form of the 'left-sided' multifractality (Mandelbrot *et al* 1990)

$$\tau(q) = \begin{cases} \text{undefined} & q < 0 \\ \text{equation (14)} & q \geqslant 0. \end{cases} \tag{15}$$

The family of scaling exponents for $q \geqslant 0$ has the similar property to the earlier analysis (de Arcangelis *et al* 1985). Thus we can extend the de Arcangelis–Redner–Coniglio model to take into account the minimum current fraction decreasing faster than power law.

We consider the hierarchical model in which the minimum current fraction decreases more slowly than the above model and decreases faster than the power law. The hierarchical model is constructed by making use of the generator shown in figure 4(a) instead of the generator (c) (or (d) or (e)) in figure 1. Similarly to the derivation of equation (3), one can derive the recursion relation of the resistances of the rare bonds between the $(n-1)$th and nth generations. The ratio of resistances of typical and rare bonds is given by

$$(R_{b,n-1}/R_{a,n-1}) = (R_{b,n}/R_{a,n}) + \tfrac{2}{9}[2 + (R_{b,n}/R_{a,n})]/[3 + (R_{b,n}/R_{a,n})]. \tag{16}$$

By renormalizing the Nth generation $(N-n)$ times, the resistance ratio is rewritten

$$(R_{b,n}/R_{a,n}) = 1 + \tfrac{2}{9}(1 + 2 + 3 + \ldots + N - n). \tag{17}$$

We consider the current fractions within the generator shown in figure 4(a). In the limit of $N \to \infty$ and $n \to 0$, the current fractions $i_{a,n}$ and $i_{b,n}$ in figure 4(a) at the nth generation are given by

$$i_{a,n} \to 1 \qquad \text{and} \qquad i_{b,n} \approx (R_{a,n}/R_{b,n}) \to 0. \tag{18}$$

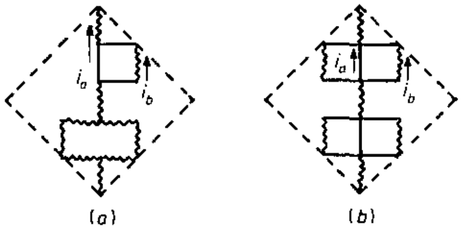


Figure 4. The generators for the rare cluster. The minimum current fraction in the hierarchical model with the generator (a) shows the different size dependence from the generators shown in figure 1(c), (d) and (e). The minimum current fraction in the model constructed by the generator (b) shows the power law.

The minimum current fraction $i_{\min}$ for the hierarchical model at the Nth generation is given by the multiplicative process on each stage, similarly to equation (11). We obtain the dependence of the minimum current fraction upon N:

$$i_{\min} \approx 9^N/(N!)^2. \tag{19}$$

By using the Stirling formula and $N = \ln L/\ln 5$, we finally obtain

$$i_{\min} \approx \exp[-(2/\ln 5)(\ln L)\ln(\ln L)]. \tag{20}$$

We can find the minimum current fraction decreasing faster than power law. However, (20) is a less decreasing minimum current fraction than (12). This shows that the behaviour of the minimum current fraction is very sensitive to the geometry of the cluster (compare (12) with (20)). This manifests itself in a large degree of variability in the minimum current fraction on different clusters of the same size.

Finally, we shall show a hierarchical fractal model in which the minimum current fraction can scale. The model is constructed by making use of the generator shown in figure 4(b) instead of the generator (c) (or (d) or (e)) in figure 1. Similarly to the derivation of (4), we obtain the ratio of resistances of typical and rare bonds

$$(R_{b,n-1}/R_{a,n-1}) = \tfrac{2}{3}(R_{b,n}/R_{a,n}) + \tfrac{4}{9}[2+(R_{b,n}/R_{a,n})]/[4+(R_{b,n}/R_{a,n})]. \tag{21}$$

In the limit of $N \to \infty$ and $n \to 0$, the above recursion relation has a finite fixed point: $(R_b/R_a)^* = (2\sqrt{10}-4)/3 = 0.775$. The current fractions in figure 4(b) are given by

$$i_a \to [2+(R_b/R_a)^*]/[4+(R_b/R_a)^*] \qquad i_b \to 1/[4+(R_b/r_a)^*]. \tag{22}$$

Thus, the minimum current fraction scales as

$$i_{\min} \approx L^{-\ln[4+(R_b/R_a)^*]/\ln 5}. \tag{23}$$

The result (23) should be compared with (12) and (20). To have the anomalous multifractal measure for current fraction distribution, it is necessary that there does not exist a finite fixed point in the recursion relation of the resistance ratio.

In summary, we present the hierarchical models which have the anomalous multifractal measure. We find that the minimum current fraction decreases faster than power law. We show that the minimum current fraction depends on the size L as follows $i_{\min} \approx \exp[-cm(\ln L)^2]$ in the hierarchical models constructed by the generator of a ladder configuration with m plaquettes. Also we find the other hierarchical model with $i_{\min} \approx \exp[-c \ln L\{\ln(\ln L)\}]$. We obtain the necessary condition that there is a faster decreasing minimum current fraction than power law.

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